5) For combined shear and biaxial edge loadings a formulation based upon the interaction formulation

$$(\sigma_x/\sigma_{x_{cr}}) + (\tau_{xy}/\tau_{xy_{cr}})^2 = 1 \quad (7a)$$

is recommended since the eigenvectors for biaxial and shear loadings are different.

Using the design technique presented in subsection IIF 1 of Ref. 2 results in designing the panel for an equivalent loading

$$\bar{N}_x = N_x[1 + \{2(N_{xy}/N_x)(K_c/C_\tau)\}^2]^{1/2}/2(N_{xy}/N_x)(K_c/C_\tau) \quad (7b)$$

where

$$C_\tau \sim [\pi^2/12(1 - \nu^2)](5.34 + 4b^2/a^2) \quad (7c)$$

is the stability constant for the simply supported plate and N_{xy} is the shear loading.

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Ablation of a Hollow Sphere

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Introduction

Citron[1] and Goodman[2] have presented approximate techniques for the solutions of ablating slabs. In this note, the two methods are adopted for determining the ablation characteristics of a hollow sphere and the results are compared.

Theoretical Analysis

A sphere with an initial outside radius a is at a constant initial temperature T_i. The inner surface, $r = b$, is insulated. The sphere is subjected to a point-symmetric, radial heat flux $Q(t)$ acting at its ablating outer surface whose radius is $r = r_s(t)$.

In the analysis, it is assumed that the heated surface remains at the melt temperature T_m, and that the molten material is immediately removed upon formation.

The premelt analysis, which is applicable during the period beginning with the initial time $t = 0$ and ending with the melt time $t = t_m$, which is the time at which the melt temperature T_m is first reached at the heated surface, has been adequately treated in the literature[3,4] and will not be pursued here.

The point-symmetric heat-conduction equation in spherical coordinates is

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2 \frac{\partial T}{\partial r}\right) \qquad b < r < r_s(t) \quad (1)$$

and the initial and boundary conditions are as follows:

$$T(r, t_m) = T_0(r, t_m) \quad (2a)$$

$$r_s(t_m) = a \quad (2b)$$

$$T(r_s, t) = T_m \quad (2c)$$

$$Q(t) = k(T_m)(\partial T/\partial r)_{r_s,t} - \rho L(dr_s/dt) \quad (2d)$$

$$(\partial T/\partial r)_{b,t} = 0 \quad (2e)$$

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where ρ, c, k, and L denote the density, specific heat, thermal conductivity, and latent heat of fusion of the material, and T_0 is the premelt temperature distribution. An auxiliary condition, $(dr_s/dt)_{t=t_m} = 0$, can be imposed from a consideration of continuity of heat input at the melt time.[1]

The solution of the posed problem will now be treated by two different approximate numerical techniques.

Method 1

In this approach, the technique utilized by Citron[1] for a slab is applied to the spherical shell. The method consists of applying a transformation which allows the consideration of a body of constant unit thickness at all times in lieu of a body of varying thickness, and then expressing the temperature distribution at any time in this unit body by a Taylor series expansion in space about the melting surface.

Using the transformation $Z = (r - b)/(r_s - b)$, and defining the following nondimensional parameters,

$$\tau = \frac{\kappa(T_m)(t - t_m)}{(a - b)^2} \qquad R_s(\tau) = \frac{a - r_s}{a - b}$$

$$\theta(Z, \tau) = \frac{T - T_i}{T_m - T_i} \qquad \bar{Q}(\tau) = \frac{Q(t)}{Q_0}$$

$$\bar{k} = \frac{k(T)}{k(T_m)} \qquad \bar{c} = \frac{c(T)}{c(T_m)}$$

$$B^* = \frac{k(T_m)[T_m - T_i]}{Q_0[a - b]} \qquad M^* = \frac{c(T_m)[T_m - T_i]}{L}$$

where Q_0 is the heat input at $t = t_m$ and κ is the thermal diffusivity, we obtain the transformed heat-conduction equation (1), which is valid for temperature-dependent material properties

$$\frac{\partial^2\theta}{\partial Z^2} = \frac{\bar{c}}{\bar{k}}(1 - R_s)^2\frac{\partial\theta}{\partial\tau} + \frac{\bar{c}}{\bar{k}}Z(1 - R_s)\dot{R}_s\frac{\partial\theta}{\partial Z} - \frac{2(1 - R_s)}{(1 - R_s)Z + \dfrac{b/a}{1 - b/a}}\frac{\partial\theta}{\partial Z} - \frac{1}{\bar{k}}\frac{d\bar{k}}{d\theta}\left(\frac{\partial\theta}{\partial Z}\right)^2 \quad (3)$$

Differentiation with respect to τ is denoted by ($\dot{\ }$). Conditions (2a–2e) become

$$\theta(Z,0) = \theta_0(Z, 0) \quad (4a)$$

$$R_s(0) = 0 \quad (4b)$$

$$\theta(1, \tau) = 1 \quad (4c)$$

$$(\partial\theta/\partial Z)_{1,\tau} = (1 - R_s)[(\bar{Q}/B^*) - (\dot{R}_s/M^*)] \quad (4d)$$

$$(\partial\theta/\partial Z)_{0,\tau} = 0 \quad (4e)$$

and $(dr_s/dt)_{t=t_m} = 0$ transforms to $\dot{R}_s(0) = 0$.

The following is a brief review of Citron's procedure: One assumes that $\theta(Z, \tau)$ can be expressed as a Taylor series expansion in space about the melting face $Z = 1$, i.e.,

$$\theta(Z, \tau) = \theta(1, \tau) + (\partial\theta/\partial Z)_{1,\tau}(Z - 1) + (\partial^2\theta/\partial Z^2)_{1,\tau}[(Z - 1)^2/2!] + \ldots \quad (5)$$

The application of condition (4e) to Eq. (5) yields

$$0 = \left(\frac{\partial\theta}{\partial Z}\right)_{1,\tau} - \left(\frac{\partial^2\theta}{\partial Z^2}\right)_{1,\tau} + \frac{1}{2!}\left(\frac{\partial^3\theta}{\partial Z^3}\right)_{1,\tau} - \ldots \quad (6)$$

where $(\partial\theta/\partial Z)_{1,\tau}$, $(\partial^2\theta/\partial Z^2)_{1,\tau}$, . . ., $(\partial^n\theta/\partial Z^n)_{1,\tau}$ can all be expressed in terms of R_s, $\dot{R}_s$, $\ddot{R}_s$, $\dddot{R}_s$, etc., by utilizing conditions (4c) and (4d), and by successively differentiating Eq. (3) with respect to Z, and evaluating the resulting expression at $Z = 1$. The substitution of these derivatives into Eq. (6) yields a nonlinear, ordinary differential equation involving R_s and its derivatives. It should be noted that this equation

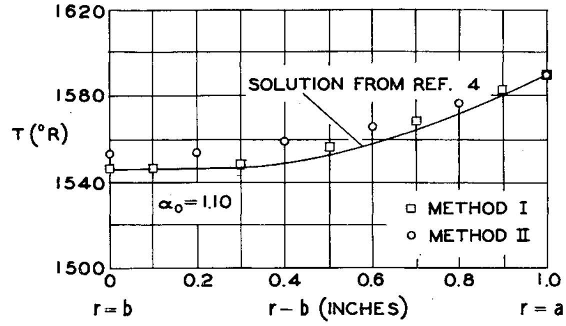

Fig. 1 Temperature profiles at melt time ($t = t_m$).

will always be linear in the highest order derivative term (when more than three terms are used in the expansion).

The series expansion of the temperature is terminated after $2K$ or $2K + 1$ terms, thus leading to a differential equation of Kth order. Thus, in addition to the initial conditions $R_s(0) = 0$, and $\dot{R}_s(0) = 0$, it is required to obtain $K - 2$ additional conditions. These $K - 2$ values, $\ddot{R}_s(0)$, $\dddot{R}_s(0)$ . . ., are obtained by matching the initial temperature distribution $\theta_0(Z, 0)$ at $K - 2$ points.

Method 2

An investigation of the heat balance integral technique for slabs is discussed by Goodman.[2] Basically, the technique is much the same as the momentum integral of fluid dynamics. The heat-conduction equation is satisfied on the average and a temperature profile is assumed. In the problem considered here, a second-degree polynomial temperature profile is assumed, and the three arbitrary constants are evaluated from the boundary conditions.

The substitution of the assumed temperature profile

$$T = T_m - \left[\frac{Q + \rho L(dr_s/dt)}{2k}\right]\left[(r_s - b) - \frac{(r - b)^2}{(r_s - b)}\right] \qquad r_s \neq b \quad (7)$$

into the integrated heat-conduction equation (1) yields the following second order, nonlinear, ordinary differential equation for $r_s(t)$:

$$\frac{d^2r_s}{dt^2} = -\left[\frac{Q}{\rho L} + \frac{dr_s}{dt}\right] \times \left\{\frac{(r_s - b)(9r_s + 7b)(dr_s/dt) + 12\kappa(3r_s - b)}{(r_s - b)^2(3r_s + 5b)}\right\} \quad (8)$$

It should be noted that Eq. (8) is applicable for temperature-independent material properties only. For materials with temperature-dependent properties, a similar procedure leads to an integrodifferential equation.

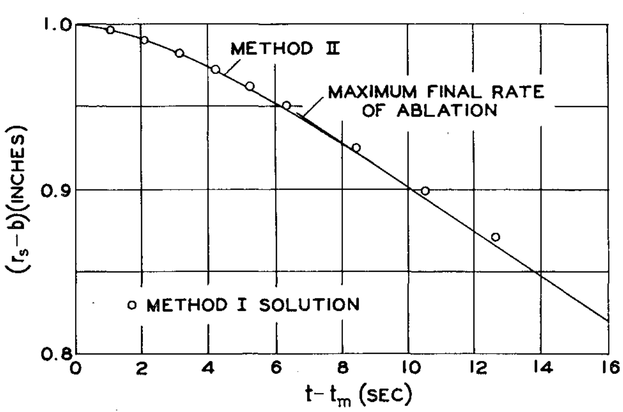

Fig. 2 Ablation depth vs time for $\alpha_0 = 1.10$.

Discussion

Numerical calculations using both methods 1 and 2 have been performed for a 1-in. thick $7\frac{3}{4}$-in. o.d. aluminum sphere subjected to a constant radial heat flux.‡ Average constant thermal and physical properties were assumed.[5] The analysis was performed for a value of $\alpha_0 = 1.10$ (where α_0 is the nondimensional surface heat-transfer coefficient ha/k) and a stagnation temperature of 1850°R.

The premelt solution for the temperature was obtained through the use of existing solutions.[4] In order to approximate the temperature profile at the melt time with sufficient accuracy, a six-term Taylor series expansion was used in method 1 (from which the required initial conditions on R_s were obtained). A comparison of the calculated temperature profile at melt time indicates that the assumed quadratic profile used in method 2 is in good agreement with both the exact premelt solution and the solution obtained from method 1 (Fig. 1).

The melt depth predicted by both methods is in excellent agreement (Fig. 2). Furthermore, as shown in Fig. 2, the rate of ablation rapidly approaches a constant value.

In conclusion, it is clearly seen that the numerically simpler technique (method 2) can be applied to the ablating sphere and yields results that are quite close to those obtained using the more complex technique (method 1). Both of these methods predict ablation profiles that are almost identical.

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‡ Since the surface temperature is assumed to remain at the melt temperature during the melt analysis, a constant heat flux corresponds to steady-state aerodynamic heating during melting.

Bow Shock Shape About a Spherical Nose

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Nomenclature

e_s	= eccentricity factor, Eq. (1)
M	= Mach number
R_N	= body nose radius
R_s	= nose radius of bow shock wave
r_s	= lateral coordinate of the bow shock wave
X	= body axial distance measured from the body nose
X'	= axial distance measured from shock apex
Δ	= stagnation-point shock standoff distance
ρ_∞/ρ_2	= density ratio across a normal shock

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